Hence, $DG = \sqrt{16r^2 - 11r^2} = r\sqrt{5}$. That is, circle LGB' is 5 times the given circle.

Also solved in various ways by J. W. BALDWIN, F. E. CANADAY, C. E. GITHENS, F. E. WOODS, J. E. HATCH, O. S. ADAMS, J. ROSENBAUM, J. M. STETSON, and J. E. MCMAHON, JR.

CALCULUS.

419. Proposed by C. C. YENN, Tangshan, North China.

Find the entire area of the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$.

SOLUTION BY THE PROPOSER.

Using the formula $\int\int \{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2\}^{1/2} dydx$, we have, from the equation of the surface,

$$\partial z/\partial x = -z^{1/3}/x^{1/3}, \qquad \partial z/\partial y = -z^{1/3}/y^{1/3};$$

whence

$$S = 8\int_0^a \int_0^{(a^{2/3}-x^{2/3})^{3/2}} \{(a^{2/3} - x^{2/3})x^{2/3} + (a^{2/3} - x^{2/3})y^{2/3} - y^{4/3}\}^{1/2} x^{-1/3} y^{-1/3} dydx. \tag{1}$$

To integrate with respect to y, let $y^2 = w^3$, so that $y^{-1/3}dy = \frac{3}{2}dw$, also put

$$A = (a^{2/3} - x^{2/3})x^{2/3}, \qquad B = (a^{2/3} - x^{2/3}). \tag{2}$$

Then

$$\int_0^{(a^{2/3}-x^{2/3})^{3/2}} \{(a^{2/3} - x^{2/3})x^{2/3} + (a^{2/3} - x^{2/3})y^{2/3} - y^{4/3}\}^{1/2} y^{-1/3} dy = \frac{3}{2}\int_0^B \{A + Bw - w^2\}^{1/2} dw$$

$$= \frac{3}{2}\left\{\frac{2w - B}{4}(A + Bw - w^2)^{1/2} + \frac{B^2 + 4A}{8}\sin^{-1}\left[\frac{2w - B}{(B^2 + 4A)^{1/2}}\right]\right\}_0^B.$$

Substituting the limits of integration, also the values of A and B from (2), we have from (1),

$$S = 6\int_0^a (a^{2/3} - x^{2/3})^{3/2}dx + 3\int_0^a (a^{2/3} - x^{2/3})(a^{2/3} + 3x^{2/3})x^{-1/3}\sin^{-1}\left\{\frac{a^{2/3} - x^{2/3}}{a^{2/3} + 3x^{2/3}}\right\}^{1/2} dx. \tag{3}$$

To evaluate the first integral, let $x = v^3$, $dx = 3v^2dv$, also put $a^{2/3} = \alpha^2$, then

$$6\int_0^a (a^{2/3} - x^{2/3})^{3/2}dx = 18\int_0^\alpha (\alpha^2 - v^2)^{3/2}v^2dv$$

$$= -18\frac{v(\alpha^2 - v^2)^{5/2}}{6}\Big|_0^\alpha + 3\alpha^2\int_0^\alpha (\alpha^2 - v^2)^{3/2}dv$$

$$= 3\alpha^2\left\{\frac{v(\alpha^2 - v^2)^{1/2}(5\alpha^2 - 2v^2)}{8} + \frac{3\alpha^4}{8}\sin^{-1}\left(\frac{v}{\alpha}\right)\right\}_0^\alpha$$

$$= \frac{9}{8}\alpha^6\frac{\pi}{2} = \frac{9\pi a^2}{16}.$$

To evaluate the second integral, put $x^{2/3} = \lambda$, $\frac{2}{3}x^{-1/3}dx = d\lambda$, also set $a^{2/3} = k$, then the second term of S in (3) becomes

$$3 \times \frac{3}{2}\int_0^k (k - \lambda)(k + 3\lambda)\sin^{-1}\left\{\frac{k - \lambda}{k + 3\lambda}\right\}^{1/2} d\lambda$$

$$= \frac{9}{2}(k^2\lambda + k\lambda^2 - \lambda^3)\sin^{-1}\left\{\frac{k - \lambda}{k + 3\lambda}\right\}^{1/2}\Big|_0^k + \frac{9}{2}\int_0^k (k^2\lambda + k\lambda^2 - \lambda^3)\frac{kd\lambda}{(k + 3\lambda)\{\lambda(k - \lambda)\}^{1/2}}, \tag{4}$$

where the first term vanishes at both limits of integration. To integrate the second term, let $\sqrt{\lambda(k - \lambda)} = \lambda\xi$, then $\lambda = k/(1 + \xi^2)$, $d\lambda = -2k\xi d\xi/(1 + \xi^2)^2$; and the second term of (4) becomes

$$-9k^3\int_\infty^0 \left(\frac{1}{1 + \xi^2} + \frac{1}{(1 + \xi^2)^2} - \frac{1}{(1 + \xi^2)^3}\right)\frac{d\xi}{\xi^2 + 4} = 9k^3 \lim_{c=\infty}\int_0^c \frac{(\xi^4 + 3\xi^2 + 1)d\xi}{(1 + \xi^2)^3(\xi^2 + 4)}.$$

Separating into partial fractions, we have

$$\int \frac{(\xi^4 + 3\xi^2 + 1)d\xi}{(1 + \xi^2)^3(\xi^2 + 4)} = \frac{5}{27}\int \frac{d\xi}{\xi^2 + 1} + \frac{4}{9}\int \frac{d\xi}{(\xi^2 + 1)^2} - \frac{1}{3}\int \frac{d\xi}{(\xi^2 + 1)^3} - \frac{5}{27}\int \frac{d\xi}{\xi^2 + 4}.$$

Applying the formula

$$\int \frac{d\xi}{(\xi^2 + a^2)^n} = \frac{1}{2(n - 1)a^2}\left\{\frac{\xi}{(\xi^2 + a^2)^{n-1}} + (2n - 3)\int \frac{d\xi}{(\xi^2 + a^2)^{n-1}}\right\} \quad [\text{here } a = 1],$$

to the second and third integrals repeatedly, simplifying and integrating, we get

$$\frac{7\xi}{72(\xi^2 + 1)} - \frac{\xi}{12(\xi^2 + 1)^2} + \frac{61}{216}\arctan \xi - \frac{5}{54}\arctan\frac{\xi}{2}.$$

Substituting the limits of integration, letting c approach ∞, and writing $a^{2/3}$ for k, we obtain

$$\frac{41}{48}\pi a^2$$

as the value of the second term of S in (3).

Hence, finally,

$$S = \frac{9\pi a^2}{16} + \frac{41\pi a^2}{48} = \frac{17}{12}\pi a^2,$$

which is the area required.

Note.—The value of S thus obtained is 34/9 [< 4] times the area of the projection of the surface on one of the coördinate planes; the latter area being $\frac{3}{8}\pi a^2$.

Also solved by E. F. CANADAY, PAUL CAPRON, and J. B. REYNOLDS.

421. Proposed by E. H. MOORE, The University of Chicago.

Given n continuous real-valued functions $\phi_g(x)$ $(g = 1, 2, \ldots, n)$ of the real variable x on the interval (01) and set $\exp \int_0^1 \phi_g(x)\phi_h(x) = w_{gh}$ $(g, h = 1, 2, \ldots, n)$. Prove that the determinant $|w_{gh}|$ of the matrix (w_{gh}) is always $\geqq 0$ and that it is > 0 if no two of the functions $\phi_1, \ldots, \phi_n$ are identically equal on (01).

SOLUTION BY C. F. GUMMER, Kingston, Ontario.

[*Note.* Since the appearance in the June issue of my "solution" for this problem. I have learned that it is based on a misinterpretation of the question. Owing probably to the uncommon use of the period in "exp." (as originally printed) I overlooked the meaning of this as a sign for the exponential function, and foolishly read it as an abbreviation for "expression." I wish to make this explanation of the error and to present my apologies to the proposer for doubts cast on the correctness of the problem. C. F. GUMMER].

Let C_{gh} denote

$$\int_0^1 \phi_g(x)\phi_h(x)dx,$$

so that $w_{gh} = \exp c_{gh}$. The theorem will be generalized by putting in the place of the exponential function any function $F(z)$ of the form $\sum_{m=0}^{\infty} a_m z^m$ having a radius of convergence greater than every c_{gh} in absolute value and having the a's, for the first of the theorem positive or zero, and for the second part all positive. Let $W_{gh} = F(c_{gh})$

It was proved in the June number (in mistake for the present theorem) that the determinant $|c_{gh}| \geqq 0$. Since the same is true for the minors of (c_{gh}) coaxial with it, (c_{gh}) is the matrix of a positive or identically vanishing quadratic form, definite or semi-definite according as $|c_{gh}| > 0$ or $= 0$. This form (in variables $y_1, \ldots, y_n$) may be written

$$\sum_{i=1}^{r}\left(\sum_{j=1}^{n} b_{ij}y_j\right)^2,$$